Exercise 1.1.6

Test for convergence

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$

Solution

Part (a)

Use the direct comparison test with the p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+n} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{converges}} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad \text{converges}$$

Part (b)

The summand can be integrated, so use the integral test. Let

$$f(x) = \frac{1}{x \ln x}.$$

x and $\ln x$ are both continuous functions for $x \geq 2$, so their product $x \ln x$ is continuous on this interval. So is their reciprocal $1/(x \ln x)$. $f(x)$ is positive for $x \geq 2$. Calculate the first derivative of $f(x)$.

$$f'(x) = \frac{d}{dx}\left(\frac{1}{x \ln x}\right) = -\frac{1 + \ln x}{x^2 (\ln x)^2}$$

$f'(x) < 0$ for all $x \geq 2$, so $f(x)$ is a monotonically decreasing function. The conditions for using the integral test are satisfied; now evaluate the corresponding integral by using the substitution $u = \ln x$ $(du = dx/x)$.

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u} = \ln u \Big|_{\ln 2}^{\infty} = \ln \infty - \ln 2 = \infty$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \quad \text{diverges.}$$

Part (c)

Use the direct comparison test with the p-series.

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}<\sum_{n=1}^{\infty}\frac{1}{n(n)}<\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\text{converges}}\quad\Rightarrow\quad\sum_{n=1}^{\infty}\frac{1}{n2^n}\quad\text{converges}$$

Alternatively, since exponents are involved, use the ratio test.

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n2^n}}&=\lim_{n\to\infty}\frac{1}{(n+1)2^{n+1}}\times\frac{n2^n}{1}\\
&=\lim_{n\to\infty}\frac{n}{(n+1)2}\\
&=\frac{1}{2}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\\
&=\frac{1}{2}\left(\frac{1}{1+0}\right)\\
&=\frac{1}{2}\\
&<1\quad\Rightarrow\quad\sum_{n=1}^{\infty}\frac{1}{n2^n}\quad\text{converges}
\end{aligned}$$

Part (d)

Notice that this is a telescoping series.

$$\begin{aligned}
\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)&=\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)\\
&=\sum_{n=1}^{\infty}[\ln(n+1)-\ln n]\\
&=(\ln 2-\ln 1)+(\ln 3-\ln 2)+(\ln 4-\ln 3)+(\ln 5-\ln 4)+\cdots
\end{aligned}$$

Since there are infinitely many terms, the sum never settles on one value. Therefore,

$$\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)\quad\text{diverges.}$$

Part (e)

Notice that

$$\begin{aligned}\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln n^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} &= \exp\left(\lim_{n\to\infty} \frac{\ln n}{n}\right) \\ &\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \exp\left(\lim_{n\to\infty} \frac{\frac{1}{n}}{1}\right) \\ &= \exp\left(\lim_{n\to\infty} \frac{1}{n}\right) \\ &= \exp(0) \\ &= 1,\end{aligned}$$

so the series

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$$

behaves as the harmonic series does as n becomes large. Consequently, it's expected to diverge. To prove this, use the direct comparison test with the series in part (b).

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}} = 1 + \sum_{n=2}^{\infty} \frac{1}{n \cdot n^{1/n}} > 1 + \underbrace{\sum_{n=2}^{\infty} \frac{1}{n \ln n}}_{\text{diverges}} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}} \quad \text{diverges}$$